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CR STRUCTURES OF CODIMENSION 2

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0. Introduction

There are several known methods of associating a Cartan connection on a principal bundle to a nondegenerate codimension 1 *CR* structure. By analyzing an appropriate moduli space, we are able to define a class of admissible codimension 2 *CR* structures which is analogous to the class of nondegenerate *CR* structures. Given an admissible *CR* structure on a manifold M we construct a principal bundle, in fact a subbundle of the frame bundle of M, and a connection on this bundle. In addition, we decompose TM as a direct sum of subbundles of fiber dimensions 1 and 2.

In §1 we present the basic definitions and some fundamental examples of *CR* structures. In §2, after defining the Levi map of a *CR* structure and the moduli space in which the Levi map is valued, we compare *CR* structures of codimensions 1 and 2 and then define admissible *CR* structures. In §3 and §4 we develop the geometry of such structures by methods reminiscent of both Chern's treatment of nondegenerate *CR* structures [1] and Webster's treatment of pseudohermitian structures [5]. In §5 we examine the moduli space in detail. Finally, in §6 we consider some examples of admissible *CR* structures, and raise a few unanswered questions. One need not master the intricacies of §5 in order to understand the discussion in §6.

Terminology is either standard or defined when it first appears. Any undefined differential geometric terms can be found in [2]. Several notational conventions are used, but not mentioned, in the text:

(1) There are a great number of indexed entities, and the range of the index varies with the entity. There are frequent reminders of the appropriate range, but there are also many instances when the range is given only implicitly, by the entity itself. The index set $\{1, 2, \cdots, n\}$ is denoted by I_n; the null set is sometimes denoted by I_0.

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(2) Summation convention is used in the sense that whenever a term involves a single letter used as both a subscript and a superscript, summation over the range appropriate to the entities involved is implied. However, when a single letter occurs more than once in a term, but only as a subscript or only as a superscript, no summation is implied. In short, $a^j b_j$ implies summation; $a^j b^j$ and $a_j b_j$ do not.

(3) If $\varphi^1, \varphi^2, \cdots, \varphi^k$ are differential forms defined on an open subset U of some manifold, then $I(\varphi^1, \varphi^2, \cdots, \varphi^k)$ denotes the ideal of differential forms on U generated by $\{\varphi^1, \varphi^2, \cdots, \varphi^k\}$.

(4) If several local sections of a bundle are under consideration, they are assumed to have the same domain unless otherwise indicated.

(5) The term *smooth* is used to mean C^∞. All manifolds and maps between them are assumed to be smooth, unless otherwise indicated.

(6) S_k is the permutation group on k letters.

(7) If E is a real vector bundle, then $\mathbb{C}E$ is its complexification.

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1. Basic definitions and an example

Definition 1. Let M be a smooth $(2n+k)$-dimensional manifold. A CR structure of dimension n and codimension k is a pair (D, J), where $D \subset TM$ is a smooth subbundle of fiber dimension $2n$ and J is a bundle automorphism of D, which satisfies the following conditions:

(a) $J^2 = -1$;

(b) if X and Y are sections of D, then $[X, Y] - [JX, JY]$ and $[JX, Y] + [X, JY]$ are also sections of D, and

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

Definition 2. Let M be a smooth $(2n+k)$-dimensional manifold. A CR structure of dimension n and codimension k is a subbundle $T'' \subset \mathbb{C}\mathbb{T}M$, of complex fiber dimension n, which satisfies the following conditions:

(a) $\overline{T''} \cap T''$ is the zero subbundle;

(b) $[T'', T''] \subset T''$.

The equivalence of Definitions 1 and 2 is easily verified. Given (D, J), extend J by complex linearity to $\mathbb{C}D$ and let

$$T'' = \{Z \in \mathbb{C}D | JZ = -iZ\}.$$

Conversely, given T'', let J_1 be the automorphism of $\overline{T''} \oplus T''$ which acts on $\overline{T''}$ (respectively T'') as multiplication by i (respectively $-i$), and then take D to be the set of real elements of $\overline{T''} \oplus T''$ and J to be the restriction of J_1 to D.

Given a CR structure in the form (D, J) (respectively T''), we shall take T'' (respectively (D, J)) to be defined in the preceding way. Also, we shall write T' for $\overline{T''}$.

Example 1. Let M be an n-dimensional complex manifold with complex structure $J: TM \to TM$. Then (TM, J) is a CR structure of dimension n and codimension 0.

Example 2. Given an $(n+1)$-dimensional complex manifold N with complex structure $J_0: TN \to TN$ and a smooth, real hypersurface $M \subset N$, let D be the maximal J_0-invariant subset of TM and denote the restriction of J_0 to D by J. It is easy to verify that (D, J) is a CR structure of dimension n and codimension 1.

Example 3. Let N be an $(n+2)$-dimensional complex manifold with complex structure $J_0: TN \to TN$. A codimension 2 submanifold M is the union of disjoint subsets M_0 and M_2 (either of which may be empty), where $p \in M_\alpha$ if and only if D_p, the maximal J_0-invariant subspace of T_pM, has real codimension α. M_2 is open. If M_2 is not empty, then $\bigcup_{p \in M_2} D_p$ is a smooth subbundle D of TM_2. By restriction, J_0 induces a bundle automorphism $J: D \to D$, and (D, J) is a CR structure of dimension n and codimension 2.

Note that if $\mathring{M}_0$, the interior of M_0 in the topology of M, is not empty, then the restriction of J_0 to $T\mathring{M}_0$ is a complex structure. Hence, by the Newlander-Nirenberg theorem, $\mathring{M}_0$ is a complex manifold. In particular, if $N = \mathbb{C}^{n+2}$ and M is compact, then, since $\mathbb{C}^{n+2}$ has no compact complex submanifolds, M_2 cannot be empty. Thus, we see that codimension 2 CR structures are in a sense unavoidable. More formally, we have proved the following theorem.

Theorem 1. *Every compact codimension 2 submanifold of $\mathbb{C}^{n+2}$ contains a nonempty open subset that inherits a CR structure of codimension 2.*

The geometry of CR structures is bound up with the algebra of hermitian forms. As a simple illustration of this connection, we consider the following examples.

Example 4. Specialize Example 2 by supposing that $N = \mathbb{C}^{n+1}$ and that M is given as the zero set of a smooth real-valued function f. Let $\hat{\theta} = J_0^* df$ and fix $q \in M$. The 2-form $d\hat{\theta}$ determines a hermitian form h_q on T'_q as follows: for all $X, Y \in T'_q$

$$h_q(X, Y) = -id\hat{\theta}(X, \overline{Y}).$$

A different defining function $\tilde{f}$ yields a different hermitian form $\tilde{h}_q$. However, it is easy to verify that $\tilde{h}_q = Ph_q$ for some $P \in \mathrm{GL}(1, \mathbb{R})$. Thus, there is an equivalence class of hermitian forms associated to q, the only invariants of which are signature and rank. These invariants determine the general nature of the local CR geometry of M.

Example 5. Specialize Example 3 by supposing that $N = \mathbb{C}^{n+2}$ and that M is given as the zero set of a pair of smooth real-valued functions (f^1, f^2). For each $\alpha \in I_2$ let $\hat{\theta}^\alpha = J_0^* df^\alpha$, and fix $q \in M$. As in Example 4, each 2-form $d\hat{\theta}^\alpha$ determines a hermitian form h_q^α on T'_q. A different pair of defining functions $(\tilde{f}^1, \tilde{f}^2)$ yields different hermitian forms $\tilde{h}_q^1$ and $\tilde{h}_q^2$. It is easy to verify that there exists $P \in \mathrm{GL}(2, \mathbb{R})$ such that $\tilde{h}_q^\alpha = P_\beta^\alpha h_q^\beta$ for all $\alpha \in I_2$. Thus, there is an equivalence class of pairs of hermitian forms associated to q, and invariants of this class are analogous to the signature and rank in the codimension 1 case.

Although Examples 4 and 5 are quite special cases, they typify the general situation insofar as the role played in the codimension 1 theory by a hermitian form is played in the codimension 2 theory by a pair of hermitian forms. Consequently, the algebraic preliminaries needed for the study of codimension 2 CR structures are much more complicated than those needed for the study of codimension 1 CR structures. An understanding of the geometric significance of these hermitian forms emerges from a more abstract study of CR structures.

2. The Levi form, the Levi map, and admissibility

For the remainder of this paper, n is a fixed positive integer and all CR structures are of dimension n.

Let T'' be a codimension k CR structure on the $(2n+k)$-dimensional manifold M, and let

$$\pi: \mathbb{C}TM \to \mathbb{C}TM/(T' \oplus T'')$$

be the natural projection. The *Levi form* of T'' is the bundle map

$$L: T' \times T' \to \mathbb{C}TM/(T' \oplus T'')$$

determined by the requirement:

$$L(X_q, Y_q) = i\pi([X, \overline{Y}]_q),$$

where X and Y are sections of T' and $q \in M$.

Let V and W be complex vector spaces, and suppose that W has a conjugation. A *W-valued hermitian form on V* is a sesquilinear map $H: V \times V \to W$ which satisfies the hermitian symmetry condition:

$$H(x, y) = \overline{H(y, x)}.$$

Given $q \in M$, let L_q denote the restriction of L to $T'_q \times T'_q$. It is easy to verify that L_q is a $\mathbb{C}T_qM/(T'_q \oplus T''_q)$-valued hermitian form on T'_q.

An *isomorphism* of two hermitian forms $H: V \times V \to W$ and $H': V' \times V' \to W'$ is a pair (A, P), where $A: V \to V'$ and $P: W \to W'$ are linear isomorphisms that satisfy the following conditions:

(a) $P(\overline{w}) = \overline{P(w)}$ for all $w \in W$ (i.e. $\overline{P} = P$);

(b) $PH(A^{-1}x, A^{-1}y) = H'(x, y)$ for all $x, y \in V'$.

T'' is *weakly uniform* if for all $p, q \in M$ the automorphism groups of L_p and L_q are isomorphic; T'' is *strongly uniform* if for all $p, q \in M$ the forms L_p and L_q are themselves isomorphic.

Let $HF^{n,k}$ be the set of all $\mathbb{C}^k$-valued hermitian forms on $\mathbb{C}^n$. $HF^{n,k}$ is a real vector space, and therefore carries a natural topology and smooth structure. The natural bases $(e_1, e_2, \cdots, e_k)$ and $(f_1, f_2, \cdots, f_n)$ of $\mathbb{C}^k$ and $\mathbb{C}^n$ induce a coordinatization of $HF^{n,k}$: the *components* of a form $H \in HF^{n,k}$ are the hermitian matrices $H^1, H^2, \cdots, H^k$ determined by the requirement that for all $r, s \in I_n$

$$H(f_r, f_s) = H^1_{r\bar{s}}e_1 + H^2_{r\bar{s}}e_2 + \cdots + H^k_{r\bar{s}}e_k.$$

The group $G^{n,k} = \mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(k, \mathbb{R})$ acts on $HF^{n,k}$ as follows:

$$(A, P) \cdot H(x, y) = PH(A^{-1}x, A^{-1}y).$$

Clearly, two forms in $HF^{n,k}$ are isomorphic if and only if they lie in the same $G^{n,k}$-orbit. Let $[HF^{n,k}]$ be the set of $G^{n,k}$-orbits and let $\rho^{n,k}: HF^{n,k} \to [HF^{n,k}]$ be the natural projection; denote the $\rho^{n,k}$-image of a subset $Y \subset HF^{n,k}$ by $[Y]$. Give $[HF^{n,k}]$ the quotient topology. Note that this topology is not Hausdorff since every neighborhood of $0 \in HF^{n,k}$ meets every $G^{n,k}$-orbit nontrivially.

The Levi form determines a map $\mathcal{L}: M \to [HF]$, called the *Levi map* of T'': for each $p \in M$ the orbit $\mathcal{L}(p)$ comprises all forms in HF that are isomorphic to L_p. The Levi map is a CR-invariant; i.e., if $\mathcal{L}_1$ and $\mathcal{L}_2$ are the Levi maps of the CR structures T_1'' and T_2'' and f is an isomorphism of T_1'' with T_2'', then $\mathcal{L}_1 = \mathcal{L}_2 \circ f$.

It is instructive at this point to examine the structure of $[HF^{n,1}]$ and its relation to the geometry of codimension 1 CR structures. Well-known results on scalar hermitian forms show that two forms in $HF^{n,1}$ are isomorphic if and only if they have the same rank and the absolute values of their signatures are equal. It is easy to verify that $[HF^{n,1}]$ is finite, but not discrete, that each orbit of nondegenerate (i.e., rank n) forms is an open one-point subset of $[HF^{n,1}]$, and that the union of these one-point subsets is a dense open subset $[Z^n] \subset [HF^{n,1}]$. A codimension 1 CR structure is *nondegenerate* if its Levi map $\mathcal{L}$ is valued in $[Z^n]$. Since $\mathcal{L}$ is continuous and $[Z^n]$ is discrete, $\mathcal{L}$ is locally constant. Therefore, the following proposition is obvious.

Proposition 1. *A nondegenerate codimension* 1 *CR structure on a connected manifold is strongly uniform.*

Let $\eta: [Z^n] \to HF^{n,1}$ be an arbitrary section; i.e., η is any map with the property that $\rho^{n,1} \circ \eta = 1$. The significance of η is that if $\mathcal{L}$ is the Levi map of a nondegenerate CR structure then $\hat{\mathcal{L}} = \eta \circ \mathcal{L}$ is a canonical smooth $HF^{n,1}$-valued map that covers $\mathcal{L}$. Clearly, $\hat{\mathcal{L}}$ is a CR-invariant that carries all the information carried by $\mathcal{L}$. Moreover, since $\hat{\mathcal{L}}$ is valued in $HF^{n,1}$ instead of $[HF^{n,1}]$, its algebraic properties are much simpler than those of $\mathcal{L}$. Although it is not usually expressed in these terms, passing from $\mathcal{L}$ to $\hat{\mathcal{L}}$ is the first step in the study of nondegenerate CR structures (see [5] and [1]). For future reference we record two properties enjoyed by any section η.

Proposition 2. (a) *For any $x \in [Z^n]$ the component of $\eta(x)$ is invertible.*

(b) *If f is a smooth map of a manifold N into $HF^{n,1}$ such that $\rho^{n,1} \circ f$ is valued in $[Z^n]$, then for each $p \in N$ there exists a smooth $G^{n,1}$-valued map g defined on some neighborhood U of p with the property that for each $q \in U$*

$$g(q) \cdot f(q) = \eta \circ \rho^{n,1} \circ f(q).$$

Proof. (a) follows immediately from the relevant definitions; (b) follows from a cursory examination of the standard procedure for diagonalizing a scalar hermitian form. q.e.d.

In order to discuss the geometry of higher-codimensional CR structures along the preceding lines, one must first gain some understanding of the moduli spaces $[HF^{n,k}]$ for $k > 1$. An obvious way to begin is to associate to each form $H \in HF^{n,k}$ the homogeneous polynomial

$$Q_H(T_1, T_2, \cdots, T_k) = \det(T_1 H^1 + T_2 H^2 + \cdots + T_k H^k),$$

and then to look for $G^{n,k}$-invariants of these polynomials. Unfortunately, the quest for such invariants is forbiddingly difficult if $k > 2$; therefore we assume that $k = 2$. To simplify notation, from now on we denote $HF^{n,2}$, $[HF^{n,2}]$, $\rho^{n,2}$ and $G^{n,2}$ by HF, $[HF]$, ρ, and G.

Given $H \in HF$, consider the inhomogeneous polynomial $P_H(T) = \det(TH^1 + H^2)$. The degree of P_H, denoted by δ_H, is not greater than n; if $\delta_H < n$, then ∞ is a root of multiplicity $n - \delta_H$. Let $R_H \subset \mathbb{P}^1(\mathbb{C})$ be the set of distinct roots of P_H. We view $\mathbb{P}^1(\mathbb{C})$ as comprising the upper half-plane $\mathbb{H}^+$, the lower half-plane $\mathbb{H}^-$, and the extended real axis $\mathbb{R} \cup \{\infty\}$ (or equivalently, $\mathbb{P}^1(\mathbb{R})$). In particular, we consider ∞ to be real.

Let $\mathcal{M}$ be the group of real Möbius transformations. Given a matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{GL}(2, \mathbb{R})$, let $\Lambda_P \in \mathcal{M}$ be the map

$$t \mapsto \frac{at + c}{bt + d}.$$

Proposition 3. *Let H and K be isomorphic forms in HF. If $K = (A, P) \cdot H$, then R_H is the Λ_P-image of R_K.*

Proof. Use the following formula:

$$P_K(T) = |\det A|^{-2} \det((aT + c)H^1 + (bT + d)H^2). \quad \text{q.e.d.}$$

For each pair of nonnegative integers (r, c) with $r + 2c = n$, let $HF(r, c)$ be the set of all $H \in HF$ such that R_H contains r real points and c pairs of complex conjugate points, and let

$$HF(n) = \bigcup_{r+2c=n} HF(r, c).$$

It follows from Proposition 3 that $HF(r, c)$ is a G-invariant subset. Using the fact that $P_H(T)$ has real coefficients, one easily proves that a form $H \in HF$ belongs to $HF(n)$ if and only if R_H contains n points, that each $HF(r, c)$ is an open subset of HF, and that $HF(n)$ is a dense open subset of HF.

We think of $[HF(n)]$ as an analogue of $[Z^n]$. Therefore, an analogue of a nondegenerate CR structure is a codimension 2 CR structure whose Levi map is valued in $[HF(n)]$. By continuity, the Levi map of such a CR structure is locally valued in some $[HF(r, c)]$; we say that it is locally of *type* (r, c). Unfortunately, the analogy breaks down when we recall Proposition 1, since the requirement that the Levi map of a codimension 2 CR structure be constant is much more stringent than that it be of some type (r, c). In order to save the analogy, we substitute weak uniformity for strong uniformity. We show in §5 that the automorphism group of a generic form in $HF(r, c)$ is conjugate to the closed Lie subgroup $G_0(r, c) \subset G$ consisting of all (A, P) such that

(a) $P = \gamma I$ for some γ in $\mathbb{R} - \{0\}$,

(b) $A = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ for some λ_i in $\mathbb{C} - \{0\}$,

(c) $|\lambda_i|^2 = \gamma$ for all i in I_r, and

(d) $\lambda_{r+2j-1}\overline{\lambda}_{r+2j} = \gamma$ for all j in I_c.

Therefore, a better analogue of a nondegenerate CR structure is a codimension 2 CR structure whose Levi map is locally valued in a suitable dense subset of some $[HF(r, c)]$. We still must find an analogue of the section $\eta: [Z^n] \to HF^{n,1}$.

Definition 1. Let r and c be nonnegative integers with $r + 2c = n$, and let $[X]$ be a subset of $[HF(r, c)]$. An *admissible section of type* (r, c) is a map $\sigma: [X] \to HF$ which satisfies the following conditions:

(a) $\rho \circ \sigma$ is the identity;

(b) for each $x \in [X]$, the automorphism group of $\sigma(x)$ is $G_0(r, c)$;

(c) for each $x \in [X]$, the components of $\sigma(x)$ are invertible;

(d) if f is a smooth map of a manifold N into HF such that $\rho \circ f$ is valued in $[X]$, then for each $p \in N$ there exists a smooth G-valued map g defined on some neighborhood U of p with the property that for each $q \in U$

$$g(q) \cdot f(q) = \sigma \circ \rho \circ f(q).$$

This definition calls for a few comments. Condition (a) merely says that σ is a section. Conditions (c) and (d) are suggested by Proposition 2; (d) is important because $[X]$ does not necessarily have a smooth structure. By the preceding discussion, if $[X]$ is generic then for each $x \in [X]$ the automorphism group of $\sigma(x)$ must be conjugate to $G_0(r, c)$; condition (b), requiring equality instead of conjugacy, simplifies matters. Note that neither (b) nor (c) is a G-invariant condition. Finally, in §5 we prove that if $n \geq 7$ then each set $[HF(r, c)]$ contains a dense open subset on which there is defined an admissible section of type (r, c) (see Theorem 5.2).

Thus, we are led to the following definition of an analogue of a nondegenerate CR structure.

Definition 2. Let σ be an admissible section of some type (r, c). A codimension 2 CR structure is *σ-admissible* if its Levi map is valued in the domain of σ.

On the surface, the requirement that a codimension 2 CR structure be admissible does not seem unduly restrictive. However, in §4 we show (see

Theorem 4.1) that if (D, J) is an admissible CR structure then D is the intersection of two globally defined distributions of rank $2n+1$ and the topology of the underlying manifold satisfies certain stringent conditions. Thus, in general, instead of hoping to find an admissible CR structure on a given manifold M, one must be content to find an admissible CR structure on some open submanifold of M (e.g., a coordinate neighborhood). We return to this point in §6, where we consider some examples.

3. The geometry of admissible CR structures: local analysis

In this section and the next, σ is an admissible section of some type (r, c), G_0 is the group $G_0(r, c)$, T'' is a σ-admissible CR structure on the $(2n+2)$-dimensional manifold M, $\mathcal{L}$ is the Levi map of T'', and $\hat{\mathcal{L}} = \sigma \circ \mathcal{L}$.

It is easy to verify that for each $q \in M$ there exists a neighborhood U of q on which a $\mathbb{C}^n$-valued 1-form ω and a $\mathbb{C}^2$-valued 1-form θ are defined such that

(a) $T''|_U$ is the kernel of θ and ω,

(b) $\mathbb{C}D|_U$ is the kernel of θ,

(c) the components of ω, $\overline{\omega}$, and θ provide a basis of complex 1-forms on U, and

(d) $\overline{\theta} = \theta$.

$(\omega; \theta)$ is called a *partially adapted moving coframe.*

The integrability condition $[T'', T''] \subset T''$ implies that each of the forms $d\theta^\alpha$ and $d\omega^j$ belongs to the ideal

$$\mathcal{I}(\theta^1, \theta^2, \omega^1, \omega^2, \cdots, \omega^n).$$

Therefore, there exist unique smooth complex functions $h^\alpha_{j\overline{k}}(\omega, \theta)$ and $a^\alpha_{jk}(\omega, \theta)$ defined on U such that

$$(1) \qquad d\theta^\alpha \equiv i h^\alpha_{j\overline{k}}(\omega, \theta)\omega^j \wedge \overline{\omega}^k + a^\alpha_{jk}(\omega, \theta)\omega^j \wedge \omega^k \mod \mathcal{I}(\theta^1, \theta^2).$$

Since $d\theta^\alpha$ is real, it follows from (1) that

$$(2) \qquad d\theta^\alpha \equiv i h^\alpha_{j\overline{k}}(\omega, \theta)\omega^j \wedge \overline{\omega}^k \mod \mathcal{I}(\theta^1, \theta^2),$$

$$(3) \qquad h^\alpha_{k\overline{j}}(\omega, \theta) = \overline{h}^\alpha_{j\overline{k}}(\omega, \theta).$$

Given $p \in U$, let $H(\omega, \theta)(p)$ be the form in HF whose components $H^\alpha(\omega, \theta)(p)$ have entries $h^\alpha_{j\overline{k}}(\omega, \theta)(p)$. A computation based on (2), (3), and the identity

$$d\theta^\alpha(X, \overline{Y}) = X\theta^\alpha(\overline{Y}) - \overline{Y}\theta^\alpha(X) - \theta^\alpha([X, \overline{Y}])$$

shows that $H(\omega, \theta)(p)$ is isomorphic to the Levi form L_p.

Proposition 1. *If* $(\omega; \theta)$ *and* $(\omega_+; \theta_+)$ *are two partially adapted moving coframes defined on the same open set* U, *then there exist smooth maps*

$$A: U \to \mathrm{GL}(n, \mathbb{C}), \quad P: U \to \mathrm{GL}(2, \mathbb{R}), \quad \textit{and} \quad v: U \to \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^n)$$

such that

(a) $\theta^\alpha = P^\alpha_\beta \theta^\beta_+$,

(b) $\omega^j = A^j_k \omega^k_+ + v^j_\alpha \theta^\alpha_+$, *and*

(c) $H(\omega, \theta) = (A, P) \cdot H(\omega_+, \theta_+)$.

Proof. (a) and (b) are obvious; (c) follows from a comparison of $d\theta$ and $d\theta_+$. q.e.d.

We now use the hypothesis that T'' is σ-admissible to define a more restricted class of moving coframes: a partially adapted moving coframe $(\omega; \theta)$ is *σ-adapted* if, for each $\alpha \in I_2$,

$$d\theta^\alpha - i\hat{\mathcal{L}}^\alpha_{j\overline{k}} \omega^j \wedge \overline{\omega}^k \equiv 0 \mod \mathcal{I}(\theta^\alpha).$$

Proposition 2. *Let* $q \in M$. *There exists a σ-adapted moving coframe defined on some neighborhood* U *of* q.

Proof. Let $(\tilde{\omega}, \tilde{\theta})$ be a partially adapted moving coframe. The map $H(\tilde{\omega}, \tilde{\theta})$ satisfies the hypothesis of condition (d) in Definition 2.1. Therefore, there exists a neighborhood U of p and smooth maps

$$A: U \to \mathrm{GL}(n, \mathbb{C}) \quad \text{and} \quad P: U \to \mathrm{GL}(2, \mathbb{R})$$

such that

$$(A, P) \cdot H(\tilde{\omega}, \tilde{\theta}) = \sigma \circ \rho \circ H(\tilde{\omega}, \tilde{\theta}).$$

Define a partially adapted moving coframe (ω_+, θ_+) on U by requiring $\omega^j_+ = A^j_k \tilde{\omega}^k$ and $\theta^\alpha_+ = P^\alpha_\beta \tilde{\theta}^\beta$. By Proposition 1,

$$H(\omega_+, \theta_+) = (A, P) \cdot H(\tilde{\omega}, \tilde{\theta}).$$

Therefore, $h^\alpha_{j\overline{k}}(\omega_+, \theta_+)$ is the restriction to U of $\hat{\mathcal{L}}^\alpha_{j\overline{k}}$.

By integrability, there exist 1-forms τ^α_β defined on U such that

$$\text{(i)} \qquad d\theta^\alpha_+ = i\hat{\mathcal{L}}^\alpha_{j\overline{k}} \omega^j_+ \wedge \overline{\omega}^k_+ + \theta^\beta_+ \wedge \tau^\alpha_\beta.$$

The forms τ^α_β determine smooth maps $a^\alpha_{\beta j}: U \to \mathbb{C}$ and $b^\alpha_{\beta\gamma}: U \to \mathbb{R}$ such that

$$\text{(ii)} \qquad \tau^\alpha_\beta = a^\alpha_{\beta j} \omega^j_+ + \overline{a}^\alpha_{\beta j} \overline{\omega}^j_+ + b^\alpha_{\beta\gamma} \theta^\gamma_+.$$

Let $\hat{\mathcal{L}}^{\alpha, \overline{k}j}$ be the functions inverse to $\hat{\mathcal{L}}^\alpha_{j\overline{k}}$; i.e.,

$$\hat{\mathcal{L}}^{\alpha, \overline{k}j} \hat{\mathcal{L}}^\alpha_{j\overline{r}} = \delta^{\overline{k}}_{\overline{r}} \quad \text{and} \quad \hat{\mathcal{L}}^\alpha_{j\overline{k}} \hat{\mathcal{L}}^{\alpha, \overline{k}s} = \delta^s_j.$$

Define new 1-forms on U by the conditions

(iii) $$\theta^\alpha = \theta^\alpha_+,$$

(iv) $$\omega^j = \omega^j_+ - i\hat{\mathcal{L}}^{2,\bar{k}j}\bar{a}^2_{1k}\theta^1 - i\hat{\mathcal{L}}^{1,\bar{k}j}\bar{a}^1_{2k}\theta^2.$$

Elementary, but lengthy, computations using (i)–(iv) show that $(\omega;\theta)$ is a σ-adapted moving coframe. q.e.d.

Proposition 3. *Let $(\omega;\theta)$ and $(\omega_+;\theta_+)$ be σ-adapted moving coframes defined on some open set U. There exist smooth maps $p: U \to \mathbb{R}$ and $a^j: U \to \mathbb{C}$ such that*

(a) *$(\mathrm{diag}(a^1, a^2, \cdots, a^n), pI)$ maps U into G_0,*

(b) *$\theta^\alpha = p\theta^\alpha_+$, and*

(c) *$\omega^j = a^j\omega^j_+$.*

Proof. Since T'' is of type (r, c), Proposition 1 implies that there exist smooth maps $p: U \to \mathbb{R}$, $a^j: U \to \mathbb{C}$, and $v^j_\alpha: U \to \mathbb{C}$ which satisfy (a), (b), and

(i) $$\omega^j = a^j\omega^j_+ + v^j_\alpha\theta^\alpha_+.$$

It follows from (a) and (b) that for each $\alpha \in I_2$

(ii) $$\mathcal{I}(\theta^\alpha) = \mathcal{I}(\theta^\alpha_+)$$

and

(iii) $$d\theta^\alpha \equiv p\,d\theta^\alpha_+ \mod \mathcal{I}(\theta^\alpha).$$

Together with the definition of a σ-adapted moving coframe, (ii) and (iii) imply that

(iv) $$\hat{\mathcal{L}}^\alpha_{j\bar{k}}\omega^j \wedge \overline{\omega}^k \equiv p\hat{\mathcal{L}}^\alpha_{s\bar{t}}\omega^s_+ \wedge \overline{\omega}^t_+ \mod \mathcal{I}(\theta^\alpha).$$

A computation using (iv), (i), and (a) shows that each v^j_α is identically zero. q.e.d.

We conclude this section by showing how a σ-adapted moving coframe determines several additional useful forms.

Proposition 4. *Let $(\omega;\theta)$ be a σ-adapted moving coframe defined on U, and suppose that*

$$\mathcal{I}^0 = \mathcal{I}(\{\theta^\alpha\}_{\alpha\in I_2}, \{\omega^j\}_{j\in I_n}),$$
$$\mathcal{I}^j = \mathcal{I}(\{\theta^\alpha\}_{\alpha\in I_2}, \{\omega^k\}_{k\in I_n-\{j\}}, \{\overline{\omega}^k\}_{k\in I_n}).$$

There exist unique 1-forms π^α and λ^j and unique 2-forms ψ^j defined on U such that

(a) $\pi^\alpha = \overline{\pi}^\alpha$,

(b) $d\theta^\alpha = i\hat{\mathcal{L}}^\alpha_{j\bar{k}}\omega^j \wedge \overline{\omega}^k + \theta^\alpha \wedge \pi^\alpha$,
(c) $d\omega^j = \omega^j \wedge \lambda^j + \psi^j$,
(d) $\psi^j \in (\mathcal{I}^j \wedge \mathcal{I}^j) \cap \mathcal{I}^0$, *and*
(e) $\frac{1}{2}(\pi^1 + \pi^2) = \frac{1}{n}\sum_{j=1}^n(\lambda^j + \overline{\lambda}^j)$.

Proof. Since $(\omega;\theta)$ is σ-adapted, there exist 1-forms π^α_+ defined on U such that

$$\text{(i)} \qquad d\theta^\alpha = i\hat{\mathcal{L}}^\alpha_{j\bar{k}}\omega^j \wedge \overline{\omega}^k + \theta^\alpha \wedge \pi^\alpha_+.$$

Let $j \in I_n$. By integrability, $d\omega^j \equiv 0 \bmod \mathcal{I}^0$. Therefore, there exist a 1-form λ^j_+ and a 2-form ψ^j_+ defined on U such that

$$\text{(ii)} \qquad d\omega^j = \omega^j \wedge \lambda^j_+ + \psi^j_+,$$

$$\text{(iii)} \qquad \psi^j_+ \in (\mathcal{I}^j \wedge \mathcal{I}^j) \cap \mathcal{I}^0.$$

Define smooth maps $b^\alpha\colon U \to \mathbb{R}$ and $c^j\colon U \to \mathbb{C}$ by the condition

$$\text{(iv)} \quad \frac{1}{2}(\pi^1_+ + \pi^2_+) - \frac{1}{n}\sum_{j=1}^n(\lambda^j_+ + \overline{\lambda}^j_+) = b^1\theta^1 + b^2\theta^2 + \sum_{j=1}^n(c^j\omega^j + \overline{c}^j\overline{\omega}^j).$$

Finally, let

$$\text{(v)} \qquad \pi^\alpha = \pi^\alpha_+ - 2b^\alpha\theta^\alpha,$$

$$\text{(vi)} \qquad \lambda^j = \lambda^j_+ + nc^j\omega^j,$$

$$\text{(vii)} \qquad \psi^j = \psi^j_+.$$

The verification of (a)–(d) is trivial; a computation based on (iv)–(vi) establishes (e). Since (b)–(d) determine ψ^j completely, determine π^α modulo $\mathcal{I}(\theta^\alpha)$, and determine λ^j modulo $\mathcal{I}(\omega^j)$, uniqueness follows from (e). q.e.d.

4. The geometry of admissible CR structures: constructions

We maintain the hypotheses and notation of the preceding section. A *σ-adapted coframe* at the point $q \in M$ is an $(n+2)$-tuple of 1-forms

$$(\omega^1(q), \omega^2(q), \cdots, \omega^n(q); \theta^1(q), \theta^2(q)),$$

where $(\omega;\theta)$ is a σ-adapted moving coframe defined on some neighborhood of q. The symbols E_q, E, and τ denote, respectively, the set of all σ-adapted coframes at q, the union $\bigcup_{q\in M} E_q$, and the natural projection of E onto M.

Let $\mathcal{U}$ be the collection of all ordered pairs $(U, (\omega; \theta))$, where $(\omega; \theta)$ is a σ-adapted moving coframe defined on the open set U. Given $x = (U, (\omega; \theta))$ in $\mathcal{U}$, let δ, p, and a^j be the component maps of $U \times G_0$. That is, for each t in $U \times G_0$

$$t = (\delta(t), (\mathrm{diag}(a^1(t), a^2(t), \cdots, a^n(t)), p(t)I)).$$

Finally, let $\Phi_x : U \times G_0 \to E|_U$ be the map

$$(a^1\delta^*\omega^1, a^2\delta^*\omega^2, \cdots, a^n\delta^*\omega^n; p\delta^*\theta^1, p\delta^*\theta^2).$$

Proposition 1. *E is a principal bundle over M with structure group G_0.*

Proof. It follows from Propositions 3.2 and 3.3 that the set of maps $\{\Phi_x | x \in \mathcal{U}\}$ defines a smooth structure on E. The remainder of the proof consists of routine verifications. q.e.d.

The *tautology forms* of T'' are 1-forms Ω^j and Θ^α defined on E as follows: given $e = (\omega(q); \theta(q))$ in E and $X \in T_eE$,

$$\Theta^\alpha(X) = \theta^\alpha(q)(\tau_* X) \quad \text{and} \quad \Omega^j(X) = \omega^j(q)(\tau_* X).$$

Theorem 1. *Consider the ideals $I_0 = \mathcal{I}(\{\Theta^\alpha\}_{\alpha \in I_2}, \{\Omega^k\}_{k \in I_n})$ and $I_j = \mathcal{I}(\{\Theta^\alpha\}_{\alpha \in I_2}, \{\Omega^k\}_{k \in I_n - \{j\}}, \{\overline{\Omega}^k\}_{k \in I_n})$. There exist unique 1-forms Π^α and Λ^j and unique 2-forms Ψ^j, all defined on E, such that*

(a) $\Pi^\alpha = \overline{\Pi}^\alpha$,

(b) $d\Theta^\alpha = i(\hat{\mathcal{L}}^\alpha_{j\bar{k}} \circ \tau)\Omega^j \wedge \overline{\Omega}^k + \Theta^\alpha \wedge \Pi^\alpha$,

(c) $d\Omega^j = \Omega^j \wedge \Lambda^j + \Psi^j$,

(d) $\Psi^j \in (I_j \wedge I_j) \cap I_0$, *and*

(e) $\frac{1}{2}(\Pi^1 + \Pi^2) = \frac{1}{n}\sum_{j=1}^n (\Lambda^j + \overline{\Lambda}^j)$.

Proof. Uniqueness is proved as in Proposition 3.4. Because of uniqueness, it suffices to prove existence locally.

Given $x = (U, (\omega; \theta))$ in $\mathcal{U}$, let π^α, λ^j, and ψ^j be the forms on U treated in Proposition 3.4. Given any differential form η on U, denote $\delta^*\eta$ by $\tilde{\eta}$. Clearly

(i) $$(\Phi_x^{-1})^*\tilde{\eta} = \tau^*\eta.$$

Finally, let $\hat{\Theta}^\alpha = (\Phi_x)^*\Theta^\alpha$ and let $\hat{\Omega}^j = (\Phi_x)^*\Omega^j$. Straightforward computations show that

(ii) $$\hat{\Theta}^\alpha = p\tilde{\theta}^\alpha,$$

(iii) $$\hat{\Omega}^j = a^j\tilde{\omega}^j,$$

(iv) $$d\tilde{\theta}^\alpha = i(\hat{\mathcal{L}}^\alpha_{j\bar{k}} \circ \delta)\tilde{\omega}^j \wedge \overline{\tilde{\omega}^k} + \tilde{\theta}^\alpha \wedge \tilde{\pi}^\alpha,$$

(v) $$d\tilde{\omega}^j = \tilde{\omega}^j \wedge \tilde{\lambda}^j + \tilde{\psi}^j.$$

It follows from (ii)–(v) and the definition of G_0 that

(vi) $$d\hat{\Theta}^\alpha = i(\hat{\mathcal{L}}^\alpha_{j\bar{k}} \circ \delta)\hat{\Omega}^j \wedge \overline{\hat{\Omega}^k} + \hat{\Theta}^\alpha \wedge (-p^{-1}\,dp + \tilde{\pi}^\alpha),$$

(vii) $$d\hat{\Omega}^j = \hat{\Omega}^j \wedge (-(a^j)^{-1}da^j + \tilde{\lambda}^j) + a^j\tilde{\psi}^j.$$

Define Π^α, Λ^j, and Ψ^j on $E|_U$ as follows:

(viii) $$\Pi^\alpha = (\Phi_x^{-1})^*(-p^{-1}dp + \tilde{\pi}^\alpha),$$

(ix) $$\Lambda^j = (\Phi_x^{-1})^*(-(a^j)^{-1}da^j + \tilde{\lambda}^j),$$

(x) $$\Psi^j = (\Phi_x^{-1})^*(a^j\tilde{\psi}^j).$$

The verifications required to complete the proof are routine. q.e.d.

Consider the following forms on E:

$$\Gamma^j = \tfrac{1}{4}(\Pi^1 + \Pi^2) + \tfrac{1}{2}(\Lambda^j - \overline{\Lambda}^j) \quad \text{for all } j \in I_r,$$
$$\Gamma^{r+2j-1} = \tfrac{1}{4}(\Pi^1 + \Pi^2) + \tfrac{1}{2}(\Lambda^{r+2j-1} - \overline{\Lambda}^{r+2j}) \quad \text{for all } j \in I_c,$$
$$\Gamma^{r+2j} = \tfrac{1}{4}(\Pi^1 + \Pi^2) + \tfrac{1}{2}(\Lambda^{r+2j} - \overline{\Lambda}^{r+2j-1}) \quad \text{for all } j \in I_c,$$
$$\Gamma^{n+\alpha} = \tfrac{1}{2}(\Pi^1 + \Pi^2) \quad \text{for all } \alpha \in I_2.$$

Finally, let $\Gamma = \operatorname{diag}(\Gamma^1, \Gamma^2, \cdots, \Gamma^{n+2})$. Straightforward computations based on (i)–(x) establish the following theorem.

Theorem 2. *Γ is a connection form on E.*

At last we can state and prove the geometric results promised in the introduction.

Let (D, J) be the real form of T''. Given $q \in M$, a σ-adapted coframe $e = (\omega(q); \theta(q))$ determines a basis

$$(\omega^1(q), \omega^2(q), \cdots, \omega^n(q); \overline{\omega}^1(q), \overline{\omega}^2(q), \cdots, \overline{\omega}^n(q); \theta^1(q), \theta^2(q))$$

of $\mathbb{C}T_q^*M$. The dual basis of $\mathbb{C}T_qM$ is of the form

$$(Z_1, Z_2, \cdots, Z_n; \overline{Z}_1, \overline{Z}_2, \cdots, \overline{Z}_n; X_1, X_2),$$

where $Z_j \in T'$ and X_α is real. Note that the real and imaginary parts of Z_j, denoted by Y_{2j-1} and Y_{2j} respectively, are in D, and that $Y_{2j} = -JY_{2j-1}$. Let

$$Q(e) = (Y_1, Y_2, \cdots, Y_{2n}, X_1, X_2).$$

The resulting map $Q\colon E \to F(M)$ is an injection; denote its image by B.

Let $\Sigma: G_0 \to \mathrm{GL}(2n+2, \mathbb{R})$ be the map that takes the matrix $(\mathrm{diag}(a^1, a^2, \cdots, a^n), pI)$ to the block diagonal matrix with jth block

$$\begin{pmatrix} \operatorname{Re} a^j & -\operatorname{Im} a^j \\ \operatorname{Im} a^j & \operatorname{Re} a^j \end{pmatrix}$$

for all $j \in I_n$, and $(n+1)$th block

$$\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}.$$

Clearly, Σ is an injection, and its image, denoted by $\tilde{G}_0$, is a closed Lie subgroup of $\mathrm{GL}(2n+2, \mathbb{R})$. We use the symbol Σ to denote not only the isomorphism between G_0 and $\tilde{G}_0$, but also the induced isomorphism between the corresponding Lie algebras $\mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_0$.

The following theorem is easily verified.

Theorem 3. *B is a $\tilde{G}_0$-structure on M, i.e., B is a smooth subbundle of $F(M)$ with structure group $\tilde{G}_0$.*

Theorem 4. *The form $\gamma = \Sigma \circ (Q^{-1})^* \Gamma$ is a connection form on B.*

Proof. Given $g \in G_0$ (respectively $\tilde{g} \in \tilde{G}_0$), let R_g (respectively $\tilde{R}_{\tilde{g}}$) denote the right action of g (respectively $\tilde{g}$) on E (respectively B). A straightforward computation yields the formula

$$Q \circ R_g = \tilde{R}_{\Sigma(g)^{-1}} \circ Q. \tag{1}$$

The verifications required to complete the proof follow from (1), Theorem 2, and the fact that the groups G_0 and $\tilde{G}_0$ are abelian. q.e.d.

Computation shows that the torsion of γ involves $\hat{\mathcal{L}}$; in particular, the torsion is not zero.

Suppose that $\tilde{T}$ is a σ-admissible CR structure on the $(2n+2)$-dimensional manifold $\tilde{M}$, with associated $\tilde{G}_0$-structure $\tilde{B}$ and connection form $\tilde{\gamma}$. Any diffeomorphism $f: M \to \tilde{M}$ induces a smooth map from $F(M)$ to $F(\tilde{M})$. We denote the restriction of this map to B by $f_\#$.

Theorem 5. *A diffeomorphism $f: M \to \tilde{M}$ is an isomorphism of T'' with $\tilde{T}''$ if and only if $f_\#$ is an isomorphism of B with $\tilde{B}$. Moreover, if f is an isomorphism, then $(f_\#)^* \tilde{\gamma} = \gamma$.*

Proof. The first statement follows from the construction of B and $\tilde{B}$. The second statement follows from the construction of γ and $\tilde{\gamma}$ and the uniqueness assertion in Theorem 1. q.e.d.

We conclude this section with the result mentioned at the end of §2.

Given $q \in M$, let $y = (Y_1, Y_2, \cdots, Y_{2n+2})$ be in B_q, let $V_j(q)$ be the subspace of T_qM spanned by $Y_{2j-1}(q)$ and $Y_{2j}(q)$ for each $j \in I_n$, and let $V_{n+\alpha}(q)$ be the subspace of T_qM spanned by $Y_{2n+\alpha}(q)$ for each $\alpha \in I_2$.

Note that for each $u \in I_{n+2}$, the space $V_u(q)$ does not depend on the choice of y. The following theorem is now obvious.

Theorem 6. *For each* $u \in I_{n+2}$, *let* $V_u = \bigcup_{q \in M} V_u(q)$.

(a) *Each set* V_u *is a smooth subbundle of* TM.

(b) $TM = V_1 \oplus V_2 \oplus \cdots \oplus V_{n+2}$.

(c) $D = V_1 \oplus V_2 \oplus \cdots \oplus V_n$.

(d) V_j *is* J*-invariant and has fiber dimension* 2 *for each* $j \in I_n$.

(e) $V_{n+\alpha}$ *has fiber dimension* 1 *for each* $\alpha \in I_2$.

Note that if V_{n+1} and V_{n+2} are trivializable, then it is possible to choose globally defined linearly independent real 1-forms θ^1 and θ^2 that annihilate D. The pair (θ^1, θ^2) is an analogue of a pseudo-hermitian structure (see [5]). The geometry of a codimension 2 CR structure, not assumed to be admissible, equipped with such a pair of 1-forms is developed in [4].

5. The moduli space $[HF]$

In this section we continue the discussion of $[HF]$ which we began after Proposition 2.2. Our main purpose is to produce examples of admissible sections; in so doing, we construct coordinates on a smooth part of $[HF]$. We assume that $n \geq 7$.

Given four points t_j in $\mathbb{P}^1(\mathbb{C})$, consider the cross ratio

$$(t_1, t_2 | t_3, t_4) = \frac{(t_1 - t_3)(t_2 - t_4)}{(t_1 - t_4)(t_2 - t_3)}$$

and let

$$((t_1, t_2 | t_3, t_4)) = \{(t_{\sigma(1)}, t_{\sigma(2)} | t_{\sigma(3)}, t_{\sigma(4)}) | \sigma \in S_4\}.$$

It is well known that this last set contains at most six elements. A form $H \in HF(n)$ is *asymmetric* if, for any subsets $S = \{s_1, s_2, s_3, s_4\}$ and $T = \{t_1, t_2, t_3, t_4\}$ of R_H, the sets $((s_1, s_2 | s_3, s_4))$ and $((t_1, t_2 | t_3, t_4))$ each contain six elements, and are disjoint unless S equals T. Let $AHF(r, c)$ be the set of asymmetric forms in $HF(r, c)$; clearly, $AHF(r, c)$ is a dense, open subset of $HF(r, c)$. Since cross ratios are preserved by Möbius transformations, it follows from Proposition 2.3 that $AHF(r, c)$ is G-invariant.

Proposition 1. *Let* H *be asymmetric. If* $\Lambda \in \mathcal{H}$ *maps* R_H *to itself, then* Λ *is the identity.*

Proof. It suffices to show that Λ fixes at least three points. By asymmetry, Λ maps each four-element subset of R_H to itself. Since R_H contains more than four points, Λ fixes each point of R_H. q.e.d.

Let $\hat{l}$ be either a vertical line in $\mathbb{C}$ or a circle in $\mathbb{C}$ centered at some point in $\mathbb{R}$; the *hyperbolic line* l is the intersection of $\hat{l}$ and $\mathbb{H}^+$. The intersection of $\mathbb{P}^1(\mathbb{R})$ and $\hat{l}$ contains exactly two points, called the *ideal points* of l, one of which is ∞ if and only if $\hat{l}$ is a vertical line. Two distinct points z_1 and z_2 in $\mathbb{H}^+$ determine a unique hyperbolic line $l(z_1, z_2)$, the ideal points of which can be labelled $p(z_1, z_2)$ and $q(z_1, z_2)$ in such a way that

$$(z_1, z_2 | q(z_1, z_2), p(z_1, z_2)) > 1;$$

the *hyperbolic distance* $d(z_1, z_2)$ is the logarithm of this cross ratio. An asymmetric form H is *regular* if R_H contains none of the ideal points of the hyperbolic lines determined by pairs of points in $R_H \cap \mathbb{H}^+$. The set of regular forms of type (r, c), denoted by $RHF(r, c)$, is a dense, open, G-invariant subset of $AHF(r, c)$. Moreover, if $c < 2$ or $r = 0$, then $RHF(r, c)$ equals $AHF(r, c)$.

We shall use the sets $RHF(r, c)$ to construct admissible sections. We begin by associating to each $H \in RHF(r, c)$ an ordered n-tuple

$$C_H = (u_1, u_2, \cdots, u_r, \beta_1, \overline{\beta}_1, \beta_2, \overline{\beta}_2, \cdots, \beta_c, \overline{\beta}_c)$$

($\langle u; \beta \rangle$ for short), where each u_i belongs to $\mathbb{R}$ and each β_i belongs to $\mathbb{H}^+$. We must distinguish four cases:

$$\text{(I) } c = 0; \quad \text{(II) } c = 1; \quad \text{(III) } c = 2; \quad \text{(IV) } c \geq 3.$$

Case I. Let $H \in RHF(r, c)$. By asymmetry and elementary properties of cross ratios, the points of R_H can be listed as $(t_1, t_2, \cdots, t_r)$ in exactly one way such that if

$$u_{j_1, j_2, j_3, j_4} = (t_{j_1}, t_{j_2} | t_{j_3}, t_{j_4}),$$

then

(a) $$u_{1,2,3,4} < u_{j_1, j_2, j_3, j_4} \quad \text{for all } \{j_1, j_2, j_3, j_4\} \neq \{1, 2, 3, 4\},$$

(b) $$u_{1,2,3,j} < u_{1,2,3,j+1} \quad \text{for all } 3 < j < r,$$

(c) $$u_{1,2,3,5} < \min\{u_{2,1,4,5}; u_{3,4,1,5}; u_{4,3,2,5}\}.$$

Let T_H be the Möbius transformation that sends $t_1 \mapsto -1$, $t_2 \mapsto 1/2$, and $t_3 \mapsto 2$, and let

$$C_H = (T_H(t_1), T_H(t_2), \cdots, T_H(t_r)).$$

Case II. Given $H \in RHF(r, c)$, write the real roots of P_H, and define T_H, as in case I, and write the complex roots of P_H as β and $\overline{\beta}$, where $T_H(\beta) \in \mathbb{H}^+$. Let

$$C_H = (T_H(t_1), T_H(t_2), \cdots, T_H(t_r), T_H(\beta), T_H(\overline{\beta})).$$

Case III. Given $H \in RHF(r,c)$, let $\{z_1, z_2\} = R_H \cap \mathbb{H}^+$, and let T_1 and T_2 be the Möbius transformations that send $p(z_1, z_2) \mapsto 0$, $q(z_1, z_2) \mapsto \infty$, and $z_1 \mapsto i$, and $p(z_2, z_1) \mapsto 0$, $q(z_2, z_1) \mapsto \infty$, and $z_2 \mapsto i$, respectively. The T_1-image and T_2-image of R_H can be uniquely written as

$$\{x_1, x_2, \cdots, x_r, i, -i, i\lambda, -i\lambda\} \quad \text{and} \quad \{y_1, y_2, \cdots, y_r, i, -i, i\lambda, -i\lambda\},$$

respectively, where $\log\lambda = d(z_1, z_2)$, and $x_1 < x_2 < \cdots < x_r$ and $y_1 < y_2 < \cdots < y_r$. (The regularity of H guarantees that no x_i or y_i is ∞.)

The map $L = T_1 \circ (T_2)^{-1}$ takes $\mathbb{H}^+$ to itself, permutes the elements of the set $\{i, \lambda, -i, -i\lambda\}$, and, since $p(z_1, z_2) = q(z_2, z_1)$, interchanges 0 and ∞. Therefore,

$$L: t \mapsto -\lambda/t.$$

Since $L = L^{-1}$, there exists a permutation $\varphi \in S_r$ such that for each $j \in I_r$ $y_j = L(x_{\varphi(j)})$ and $x_{\varphi(j)} = L(y_j)$. In particular, $y_1 = L(x_{\varphi(1)})$ and $x_{\varphi(1)} = L(y_1)$. Therefore, if $x_1 = y_1$, then L maps the set $\{i, i\lambda, -i, -i\lambda, x_{\varphi(1)}, x_1\}$ to itself, and by the proof of Proposition 1, L is the identity—a contradiction. If $x_1 < y_1$, let $T_H = T_1$ and

$$C_H = (x_1, x_2, \cdots, x_r, i, -i, i\lambda, -i\lambda);$$

if $y_1 < x_1$, let $T_H = T_2$ and

$$C_H = (y_1, y_2, \cdots, y_r, i, -i, i\lambda, -i\lambda).$$

Case IV. Given $H \in RHF(r,c)$, let z_j and z_k (respectively $z_{j'}$ and $z_{k'}$) be distinct points of $R_H \cap \mathbb{H}^+$, separated by the hyperbolic distance d (respectively d'). If $d = d'$, then $\{z_j, z_k\} = \{z_{j'}, z_{k'}\}$. Indeed, let $p = p(z_j, z_k)$ and $q = q(z_j, z_k)$, and let L be the Möbius transformation that sends $p \mapsto 0$, $q \mapsto \infty$, and $z_j \mapsto i$. It follows from the definition of p and q that $(z_j, z_k | q, p) > 1$. Since L preserves cross ratios, $L(z_k) = is$ for some $s > 1$; since L is an isometry, $d = \log s$. Define L' and s' similarly, using j' and k' in place of j and k. The assumed equality of d and d' implies the equality of s and s'. Let $\tilde{L} = (L')^{-1} \circ L$, and note that $\tilde{L}$ maps z_j to $z_{j'}$ and z_k to $z_{k'}$. Therefore

$$(z_j, z_k | \overline{z}_j, \overline{z}_k) = (z_{j'}, z_{k'} | \overline{z}_{j'}, \overline{z}_{k'}),$$

so, by asymmetry, $\{z_j, z_k\} = \{z_{j'}, z_{k'}\}$.

There is a unique way to write the roots of P_H as

$$(x_1, x_2, \cdots, x_r, z_1, \overline{z}_1, z_2, \overline{z}_2, \cdots, z_c, \overline{z}_c)$$

such that

(a) $d(z_1, z_{j+1}) < d(z_1, z_{j+2})$ for all j in I_{c-2},

(b) $d(z_1, z_3) < d(z_2, z_{2+j})$ for all j in I_{c-2}, and

(c) $T_H(x_j) < T_H(x_{j+1})$ for all j in I_{r-1}, where T_H is the Möbius transformation that sends $p(z_1, z_2) \mapsto 0$, $q(z_1, z_2) \mapsto \infty$, and $z_1 \mapsto i$. Let $C_H = \langle x; z \rangle$.

Proposition 2. *If H and K are isomorphic regular forms, then $C_H = C_K$.*

Proof. Suppose that $K = (A, P) \cdot H$. By Proposition 2.3, R_H is the Λ_P-image of R_K; since C_H and C_K are constructed from R_H and R_K by devices that are invariant under the action of $\mathcal{H}$, it follows that $C_H = C_K$. q.e.d.

A form H is *quasi-canonical* if it is regular and C_H is a list of the points of R_H.

Proposition 3. *Let H be quasi-canonical. Then R_H contains neither* 0 *nor ∞, and H has invertible components.*

Proof. Suppose that H is of type (r, c) and that $C_H = \langle x; z \rangle$. If $c \leq 1$, then $x_1 = -1$, $x_2 = 1/2$, and $x_3 = 2$. It follows that if x_j is 0 or ∞, then $((x_1, x_2 | x_3, x_j))$ contains only three elements, which contradicts the asymmetry of H.

If $c > 1$, then $z_1 = i$ and $z_2 = i\lambda$, and the ideal points of $l(z_1, z_2)$ are 0 and ∞. The regularity of H implies that neither 0 nor ∞ is in R_H. Finally, note that H^1 (respectively H^2) is invertible if and only if ∞ (respectively 0) is not a root of P_H. q.e.d.

We need some additional terminology and notation. Let $J(r, c)$ be the set of all

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r; \varepsilon_{r+1}, \varepsilon_{r+2}, \cdots, \varepsilon_{r+2c}),$$

where $|\varepsilon_j| = 1$ for all j in I_n and $\varepsilon_j = 1$ if $j > r$ or $j = 1$. If $\varepsilon \in J(r, c)$, $u_i \in \mathbb{R}$ for each i in I_r, and $\beta_j \in \mathbb{C}$ for each j in I_c, then $M(u; \beta; \varepsilon)$ is the $n \times n$ block diagonal hermitian matrix with jth block $\varepsilon_j u_j$ for each j in I_r and $(r + j)$th block

$$\begin{pmatrix} 0 & \beta_j \\ \overline{\beta}_j & 0 \end{pmatrix}$$

for each j in I_c. If each u_i and β_j equals 1, then we write $M(1; 1; \varepsilon)$ for $M(u; \beta; \varepsilon)$.

Given $H \in RHF(r, c)$, let $C_H = \langle u; \beta \rangle$. H is *canonical* if it is quasi-canonical and if for some $\varepsilon \in J(r, c)$, $H^1 = -M(1; 1; \varepsilon)$ and $H^2 = M(u; \beta; \varepsilon)$.

Theorem 1. *Let $H \in RHF(r, c)$. H is isomorphic to a unique canonical form H_0. The automorphism group of H_0 is $G_0(r, c)$. Moreover, H has a neighborhood $U \subset RHF(r, c)$ on which there is defined a smooth G-valued map g such that $K_0 = g(K) \cdot K$ for each $K \in U$.*

Proof. By construction, C_H is the T_H-image of a listing of the points of R_H for some $T_H \in \mathcal{H}$. Choose $P \in \mathrm{GL}(2,\mathbb{R})$ so that $(\Lambda_P)^{-1} = T_H$, and let $H' = (I,P)\cdot H$. By Proposition 2.3, $R_{H'}$ is the T_H-image of R_H; hence, C_H is a listing of the points of $R_{H'}$. Since $C_{H'} = C_H$ by Proposition 2, H' is quasi-canonical.

We now suppose that H is quasi-canonical and shall show that H is isomorphic to a canonical form. Let $C_H = (t_1, t_2, \cdots, t_n)$. By Proposition 3, H^1 is invertible; it follows from the definition of P_H that $t_1, t_2, \cdots, t_n$ are distinct eigenvalues of the matrix $-H^2(H^1)^{-1}$. For each $i \in I_n$ choose a row eigenvector $(E_i)^T$ with eigenvalue t_i. Clearly

$$(E_i)^T(t_i H^1 + H^2) = 0.$$

Since H^1 and H^2 are hermitian, conjugation and transposition yield

$$(\bar{t}_i H^1 + H^2)\overline{E}_i = 0.$$

Consequently, for any $j,\ k \in I_n$

$$(E_j)^T(t_j H^1 + H^2)\overline{E}_k = 0 \quad \text{and} \quad (E_j)^T(\bar{t}_k H^1 + H^2)\overline{E}_k = 0.$$

Therefore, if $t_j \neq \bar{t}_k$, then

$$(E_j)^T H^1 \overline{E}_k = 0 \quad \text{and} \quad (E_j)^T H^2 \overline{E}_k = 0$$

and $H(E_j, E_k) = 0$.

If $r = 0$, let $\delta_1 = 1$. Otherwise, for each j in I_r,

$$H(E_j, E_j) = r_j^1 e_1 + r_j^2 e_2 \neq 0,$$

where r_j^1 and r_j^2 are real, and $r_j^1 t_j + r_j^2 = 0$. Let $s_j = |r_j^1|^{-1/2}$, let $\delta_j = r_j^1/|r_j^1|$, and let $\varepsilon_j = \delta_j \delta_1$.

If $c > 0$, then for each j in I_c,

$$H(E_{r+2j-1}, E_{r+2j}) = \gamma_j^1 e_1 + \gamma_j^2 e_2 \neq 0,$$

where $\gamma_j^1 t_{r+2j-1} + \gamma_j^2 = 0$. Let $s_{r+2j-1} = \delta_1/\gamma_j^1$, let $s_{r+2j} = 1$, and let $\varepsilon_{r+2j-1} = \varepsilon_{r+2j} = 1$.

The map that takes $s_j E_j$ to f_j for each j in I_n determines a matrix A in $\mathrm{GL}(n,\mathbb{C})$. Let $H_0 = (A, -\delta_1 I)\cdot H$. It is easy to verify that H_0 is canonical. Indeed, $(H_0)^1 = -M(1;1;\varepsilon)$ and $(H_0)^2 = M(u;\beta;\varepsilon)$, where ε is as above, $u_j = t_j$ for each $j \in I_r$, and $\beta_j = t_{r+2j-1}$ for each $j \in I_c$.

Thus, we have proved that every regular form is isomorphic to a canonical form H_0. In order to show that H_0 is unique, we shall prove that two canonical forms are isomorphic if and only if they are equal.

Let H and K be canonical forms of type (r,c) and suppose that $K = (A,P)\cdot H$. By Proposition 2.3, R_H is the Λ_P-image of R_K; by Proposition

2, $C_H = C_K$. Since C_H and C_K are lists of the points of R_H and R_K, respectively, it follows that Λ_P maps R_H to itself. Hence, by Proposition 1, Λ_P is the identity, so $P = pI$ for some $p \in \mathbb{R} - \{0\}$. Let $C_H = \langle u; \beta \rangle$. For some choice of ε and $\tilde{\varepsilon}$ in $J(r,c)$,

$$H^1 = -M(1;1;\varepsilon), \quad H^2 = M(u;\beta;\varepsilon),$$
$$K^1 = -M(1;1;\tilde{\varepsilon}), \quad K^2 = M(u;\beta;\tilde{\varepsilon}).$$

It is easy to verify that A is a diagonal matrix. Let a_j be the jth diagonal entry of A. Then, for each $j, k \in I_n$

$$\text{(i)} \qquad a_j\overline{a}_k K(f_j, f_k) = pH(f_j, f_k).$$

If $c \neq 0$, then (i) implies that for each $j \in I_c$

$$\text{(ii)} \qquad a_{r+2j-1}\overline{a}_{r+2j} = p.$$

If $r \neq 0$, then for each $j \in I_r$

$$H(f_j, f_j) = \varepsilon_j(-e_1 + u_j e_2) \quad \text{and} \quad K(f_j, f_j) = \tilde{\varepsilon}_j(-e_1 + u_j e_2).$$

Therefore, (i) implies that $|a_j|^2\tilde{\varepsilon}_j = p\varepsilon_j$. Since $\tilde{\varepsilon}_1 = \varepsilon_1 = 1$ by definition, $p > 0$. Since $|\tilde{\varepsilon}_j| = |\varepsilon_j| = 1$, it follows that $\tilde{\varepsilon}_j = \varepsilon_j$ and

$$\text{(iii)} \qquad |a_j|^2 = p.$$

Thus, $H = K$.

It follows from (ii) and (iii) that the automorphism group of a canonical form of type (r,c) is $G_0(r,c)$. Therefore, all that remains is to verify the assertion of smoothness. Since this verification is straightforward, but tedious, we shall omit it. q.e.d.

It follows from Theorem 1 that if H and K are isomorphic regular forms of type (r,c) then $H_0 = K_0$. Therefore

$$\sigma_{r,c}\colon [H] \mapsto H_0$$

is a well-defined map from $[RHF(r,c)]$ to $RHF(r,c)$.

Theorem 2. (a) *$\sigma_{r,c}$ is an admissible section.*

(b) *The set $[RHF] = \bigcup_{r+2c=n}[RHF(r,c)]$ is a dense open subset of $[HF]$.*

Proof. (a) All of the required verifications are simple consequences of Theorem 1.

(b) We already know the following facts:

(i) $HF(n)$ is a dense open subset of HF;
(ii) $HF(n) = \bigcup_{r+2c=n} HF(r,c)$;
(iii) $AHF(r,c)$ is a dense open subset of $HF(r,c)$;
(iv) $RHF(r,c)$ is a dense open subset of $AHF(r,c)$.

Therefore, it is clear that $RHF = \bigcup_{r+2c=n} RHF(r,c)$ is a dense open subset of HF, which, together with the definition of the quotient topology, implies the desired result. q.e.d.

As we shall see, Theorem 2 is useful in producing examples of admissible CR structures.

6. Examples of admissible CR structures and final remarks

Example 1. Return to Example 1.5. The forms h_q^1 and h_q^2 are hermitian forms on T_q'. Choose a basis of T_q', let H^1 and H^2 be the matrices of h_q^1 and h_q^2 relative to this basis, and let $H \in HF$ be the form whose components are H^1 and H^2. It follows from the construction of h_q^1 and h_q^2 and the discussion of partially adapted moving coframes at the beginning of §3 that $[H] = \mathcal{L}(q)$, where $\mathcal{L}$ is the Levi map of T''. If H belongs to some $RHF(r,c)$, then by continuity $\mathcal{L}$ maps some neighborhood of q into $[RHF(r,c)]$; by definition, the CR structure on this neighborhood is $\sigma_{r,c}$-admissible. Moreover, this argument together with Theorem 5.2(b) shows that if $H \notin RHF$ then it is possible to deform some neighborhood of q into an admissible CR submanifold of $\mathbb{C}^{n+2}$.

In the preceding example, it is not hard to write down formulas for partially adapted moving coframes in terms of the defining functions. However, if T'' is $\sigma_{r,c}$-admissible, explicit formulas for $\sigma_{r,c}$-adapted moving coframes are in general unobtainable, since the map $\hat{\mathcal{L}} = \sigma_{r,c} \circ \mathcal{L}$ depends on the roots of certain polynomials. (See the construction of $\sigma_{r,c}$ in §5.) Nonetheless, there is a class of admissible CR structures where everything can be written down explicitly.

Example 2. Let $(x^1 + iy^1, x^2 + iy^2, z^1, z^2, \cdots, z^n)$ denote a point in $\mathbb{C}^{n+2}$, and let $J\colon T\mathbb{C}^{n+2} \to T\mathbb{C}^{n+2}$ be the natural complex structure.

Given $H \in HF$, for each $\alpha \in I_2$ let

$$\hat{\varphi}^\alpha = dy^\alpha - \tfrac{1}{2}H^\alpha_{j\bar{k}} z^{\bar{k}} dz^j - \tfrac{1}{2}H^\alpha_{j\bar{k}} z^j dz^{\bar{k}},$$

$$\hat{\theta}^\alpha = dx^\alpha - \tfrac{i}{2}H^\alpha_{j\bar{k}} z^{\bar{k}} dz^j + \tfrac{i}{2}H^\alpha_{j\bar{k}} z^j dz^{\bar{k}}.$$

Note that $(\hat{\varphi}^1, \hat{\varphi}^2, \hat{\theta}^1, \hat{\theta}^2, dz^1, dz^2, \cdots, dz^n, dz^{\bar{1}}, dz^{\bar{2}}, \cdots, dz^{\bar{n}})$ is a global basis for $\mathbb{C}T^*\mathbb{C}^{n+2}$.

For each $\alpha \in I_2$, let $f^\alpha = y^\alpha - \frac{1}{2}H^\alpha_{j\bar{k}} z^j z^{\bar{k}}$ and let M be the zero set of f^1 and f^2. Since $df^\alpha_p = \hat{\varphi}^\alpha_p$ and $J^* df^\alpha_p = \hat{\theta}^\alpha_p$ for each $p \in M$, M is a smooth codimension 2 submanifold of $\mathbb{C}^{n+2}$ and M inherits a codimension 2 CR structure T'' from $\mathbb{C}^{n+2}$. Thus, we have a special case of Example

1. Moreover, it is easy to write down a partially adapted moving coframe (ω, θ): let $\omega^j = dz^j|_M$ and $\theta^\alpha = \hat{\theta}^\alpha|_M$. Since $d\theta^\alpha = iH^\alpha_{j\bar{k}}\omega^j \wedge \omega^{\bar{k}}$, the Levi map of T'' is the constant map $\mathcal{L}: p \mapsto [H]$. Suppose that $H \in RHF(r, c)$, choose $(A, P) \in G$ so that $(A, P) \cdot H = \sigma_{r,c}([H])$, and let

$$\tilde{\omega}^j = A^j_k \omega^k \quad \text{for all } j \in I_n \qquad \text{and} \qquad \tilde{\theta}^\alpha = P^\alpha_\beta \theta^\beta \quad \text{for all } \alpha \in I_2.$$

Then T'' is $\sigma_{r,c}$-admissible, and $(\tilde{\omega}, \tilde{\theta})$ is a $\sigma_{r,c}$-adapted moving coframe.

In the theory of codimension 1 *CR* structures, one often considers a hyperquadric as a homogeneous model space for a nondegenerate *CR* structure (see [1]). It is possible to show that the quadrics in Example 2 are also homogeneous spaces. However, since their *CR* structures are strongly uniform, while, in general, an admissible *CR* structure is only weakly uniform, it is not clear that these quadrics can be of any particular use.

We conclude by mentioning several problems that might prove to be interesting:

(1) Determine whether a given set of functions defined on some open set in $\mathbb{R}^{2n+2}$ can arise as the functions $\hat{\mathcal{L}}^\alpha_{j\bar{k}}$ determined by an admissible *CR* structure. This problem is somewhat similar to the problem of prescribing curvature in Riemannian geometry, and, not surprisingly, leads to a complicated system of nonlinear PDE. Since the Levi map of a nondegenerate *CR* structure is locally constant, there is no strictly analogous problem in codimension 1.

(2) Investigate the relationship between the topology of a manifold and the possibility of its carrying a globally defined admissible *CR* structure. Theorem 4.6 shows that the topology must be fairly simple if an admissible *CR* structure is to exist. It would be interesting to know if there are compact manifolds that carry admissible *CR* structures.

(3) Find a weaker notion of admissibility that is still strong enough to yield geometric results. Theorem 5.2 suggests that admissibility is not unduly restrictive; Theorem 4.6 suggests the contrary. If the theory in codimension 1 is a trustworthy guide, then some sort of restriction on the Levi map is probably necessary. One possibility is to require the Levi map of the *CR* structure to be valued in some $[HF(r, c)]$. However, such a *CR* structure will not generally be weakly uniform, so the geometric methods of this paper are inapplicable.

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